Donaldson = Seiberg-Witten from Mochizuki's formula and instanton counting

Kaleidoscopic View of Modern Mathematics

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Joint work with Lothar Göttsche, Kota Yoshioka,
relying on the work of Takuro Mochizuki

www.kurims.kyoto-u.ac.jp/~nakajima/Talks/2011-01-10 Paris.pdf

In this lecture,

X : compact, oriented, C^∞ 4-mfd with $b^+ > 1$, $b_1 = 0$

Let $\chi(X)$ = Euler #, $\sigma(X)$ = signature.

For convieniece, let

$$(K_X^2) := 2\chi(X) + 3\sigma(X), \quad \chi_h(X) := \frac{\chi(X) + \sigma(X)}{4}$$

If X : complex projective surface, $\Longrightarrow \begin{cases} (K_X^2) = \text{self-intersection number of } K_X \\ \chi_h(X) = \text{holomorphic Euler characteristic} \end{cases}$

These are classical invariants of X.

We have two gauge theoretic invariants of X,
defined by the following recipe:

— Take a Riemannian metric g on X.
— Consider the space of (equivalence classes of) solutions of nonlinear partial differential equations,
—— moduli spaces.
— Integrate certain natural cohomology classes over moduli spaces.

① Donaldson invariants (1989)
…… an infinite sequence of C^∞-invariants of X, defined via moduli spaces of $SO(3)$-instantons

② Seiberg-Witten invariants (1994)
…… an invariant of a spin^c-structure on X
(zero except finitely many spin^c-structures), defined via moduli spaces of $U(1)$-monopoles

Witten conjecture (1994)

generating function of Donaldson invariants

$=$ generating function of SW-invariants

Under a technical assumption : X: simple type

$$\mathcal{D}^{\mathfrak{z}}(\exp\alpha\,(1+\tfrac{1}{2}p)) = (-1)^{\chi_h(X)}\,2^{(K_X^2)-\chi_h(X)+2}\,e^{(\alpha^2)/2}\sum_{\substack{\mathfrak{s}:\,\mathrm{spin}^c\text{ str}\\ (\text{finite sum})}} SW(\mathfrak{s})\,(-1)^{(\mathfrak{z},\,\mathfrak{z}+c_1(\mathfrak{s}))/2}\,e^{(c_1(\mathfrak{s}),\,\alpha)}$$

$\mathfrak{z} \in H^2(X;\mathbb{Z})$ (fixed)

$\alpha \in H_2(X;\mathbb{R})$, $p =$ pt class $\in H_0(X;\mathbb{R})$

I do not explain details of this formula. But it is striking:

- Moduli spaces of instantons and monopoles are close cousins, but no direct relations.

— In fact, monopoles are much easier to deal with, than instantons

- In LHS, infinitely many $SO(3)$-instanton moduli spaces with various p_1 are used.

$$\mathcal{D}^{\xi}\left(\exp(\alpha z)\left(1+\tfrac{1}{2}p\right)\right)=\sum_{n}\int_{M(2,3,n)}\exp(\mu(\alpha))\left(1+\tfrac{1}{2}\mu(p)\right)$$ ← formal infinite sum

(Witten conjecture does not make sense for individual moduli spaces.)

Witten's argument was based on Seiberg–Witten ansatz for the $N=2$ SUSY Yang–Mills theory.

<u>Witten</u> (1988)

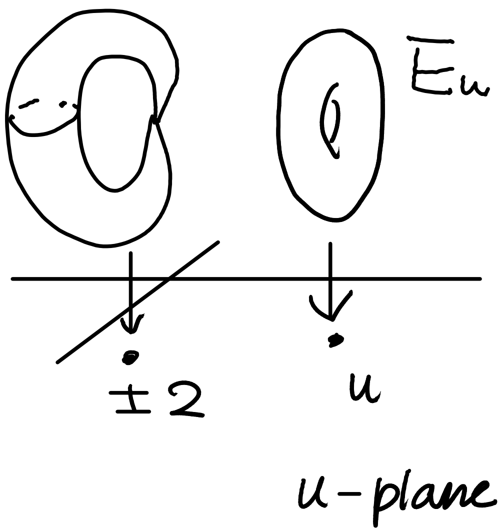
partition function = Donaldson invariants

path integral over $\mathcal{B}$: the space of all connections + various fields

<u>SW</u> (1994)

This theory (for $X=\mathbb{R}^4$) is "controlled" by a family of elliptic curves :

$E_u : y^2 = 4x(x^2 + ux + 1)$ singular at $u = \pm 2$



{ Donaldson invariants and SW invariants appear as semiclassical expansions of the same quantum field theory at different points { $u=\infty$ and $u=\pm 2$

u (the parameter for the theory) is originally defined by expansion at

- $u = \infty$
- $u = \pm 2$ $\longrightarrow$ It is a formal power series.

— it is analytically continued to the whole plane.
— the theory depends holomorphically on u
— it has modular invariance.

At the present time, no one can justify these in a mathematically rigorous way directly.

But we <u>do</u> know answers to several natural questions :

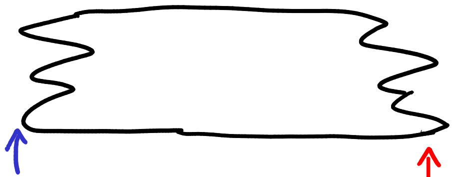
1) How does the partition function defined for $X = \mathbb{R}^4$?
— Nekrasov (2002) gave a rigorous definition, using the equivariant homology group.

2) Why it is something to do with elliptic curves ?
— Nekrasov conjecture proved by
Nekrasov - Okounkov, N - Yoshioka, Braverman – Etingof.
This is a kind of "mirror symmetry".
The proofs are "computation".
..... Not so satisfactory, but at least rigorous .....

(u = generating function of certain equivariant integrals over (framed) moduli space of $SO(3)$-instantons on $\mathbb{R}^4$)

The remaining question : How to apply our knowledge for $X = \mathbb{R}^4$ to general X ?

Pidstrigach–Tyurin, Feehan–Leness proposed a rigorous approach using moduli spaces of SO(3)-monopoles as a cobordism between



U(1)-monopole moduli and SO(3)-instanton moduli.

Thus they connected SW and Donaldson invariants in a classical field theory.

Application

[FL] showed (under a certain technical assumption)

① $$D^{\mathfrak{z}}\left(\exp(\alpha z)(1+\tfrac{1}{2}p)\right) = \sum_{\mathfrak{s}} f(\chi_h(X), (K_X^2), \mathfrak{s}, \mathfrak{z}, \alpha, \mathfrak{s}_0) \times SW(\mathfrak{s})$$

auxiliary spin^c str

② Witten's conjecture is true if $(K_X^2) \geq \chi_h(X) - 3$ or X: projective surface

The coefficients f, defined via $SO(3)$-monopole moduli spaces, are difficult to compute explicitly.

Also the role of the SW curve $y^2 = 4x(x^2+ux+1)$ is not clear in this approach.

Mochizuki (2009, preliminary version exists from 2002)
Assume X : cpx projective surface

① Replace moduli spaces of $SO(3)$-monopoles by
their algebro-geometric counter parts :
moduli space of pairs of torsion free sheaves and their sections

② Define virtual fundamental classes on them
(like the case of Gromov-Witten invariants)

③ Apply virtual fixed point formula :
$\Longrightarrow$ The coefficients f are now replaced by
explicit integrals over Hilbert scheme of points on X
----- nice resolution of $S^nX = X^n/\mathfrak{S}_n$

But still need to identify with those in Witten's conjecture.

Problem : Develop the same theory for C^∞ 4-manifolds
(for almost cpx surfaces).

<u>GNY</u> : Computation of the integral.

1°. enough to compute for X : toric surface

····· not trivial, but well-known since the work of Ellingsrud-Göttsche-Lehn

2°. fixed point formula $\Longrightarrow$ enough to compute for "$X = \mathbb{R}^4$"!

$\exists$ combinatorial formula

but still difficult to compute since # of pts is arbitrary

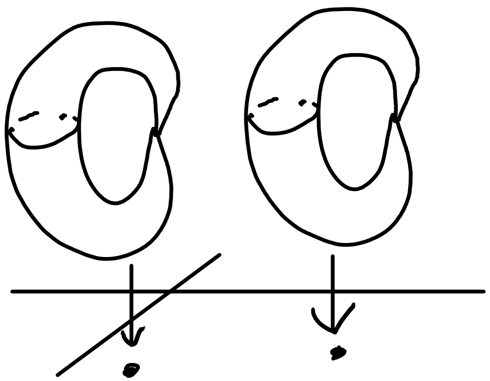
The integral for $X = \mathbb{R}^4$ is a variant of Nekrasov partition function.

($N=2$ SUSY YM theory with a fundamental matter)

$\Longrightarrow$ can be written in terms of SW curve for the theory with matter:

$$y^2 = 4x^2(x+u) + 4mx + 1$$

The additional parameter m (matter) is specialised so that it is a family of degenerate elliptic curves.



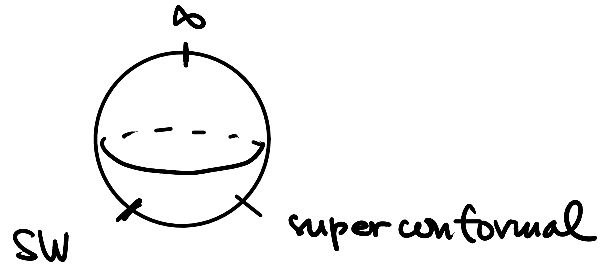
We get a different SW curve, since we have studied $SO(3)$-monopoles.

the integral $=$ residue of an explicit differential ⑧
at $u=\infty$

which extends to meromorphically to $\mathbb{P}^1$

This is close to Witten's intuition.
But now we get a new feature:

— It has 3 poles
a) $u=\infty$, b) SW contribution



and c) superconformal point [Mariño-Moore-Peradze]
($\Leftrightarrow$ both A & B cycles collapse)

So by Residue thm $\Rightarrow$ $\underset{u=\infty}{\mathrm{Res}} + \underset{SW}{\mathrm{Res}} + \underset{s.c.\,pt}{\mathrm{Res}} = 0$

$\Downarrow$ $\mathcal{D}$ by Th ①

$\Downarrow$ The RHS of Witten's conjecture

$\Downarrow$ a possible new contribution?

Ref (Mariño-Moore-Peradze)

Assume X: SW simple type. We say X is of superconformal simple type

$\underset{\text{def.}}{\iff}$ a) $(K_X^2) \geq \chi_h(X) - 3$

or b) $$\sum_{\mathfrak{s}} (-1)^{(K_X, K_X + c_1(\mathfrak{s}))/2} SW(\mathfrak{s}) (c_1(\mathfrak{s}), \alpha)^n = 0$$

$$0 \leq \overset{\forall}{n} \leq \chi_h(X) - (K_X^2) - 4$$

Finally

obviously true

Donaldson inv. depends only on $\mathfrak{z}$ mod 2 (up to sign)

$\Rightarrow X$: superconformal simple type.

$\Rightarrow \sum_{\mathfrak{s}} SW(\mathfrak{s}) \otimes$ is regular at superconformal pt.

Hence

Residue Thm $\Longrightarrow$ Witten's conjecture is true.